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A new technique for the link slice problem

Michael H. Freedman*

University of California, San Diego, La Jolla, CA 92093, USA

The conjectures that the 4-dimensional surgery theorem and 5-dimensional s-cobordism theorem hold without fundamental group restriction in the topological category are equivalent to assertions that certain "atomic" links are slice. This has been reported in [CF, F2, F4 and FQ]. The slices must be topologically flat and obey some side conditions. For surgery the condition is: $\pi_1(S^3 - \partial\,\text{slice}) \to \pi_1(B^4 - \text{slice})$ must be an epimorphism, i.e., the slice should be "homotopically ribbon"; for the s-cobordism theorem the slice restricted to a certain trivial sublink must be standard. There is some choice about what the atomic links are; the current favorites are built from the simple "Hopf link" by a great deal of Bing doubling and just a little Whitehead doubling. A link typical of those atomic for surgery is illustrated in Fig. 1. (Links atomic for both s-cobordism and surgery are slightly less symmetrical.)

There has been considerable interplay between the link theory and the equivalent abstract questions. The link theory has been of two sorts: algebraic invariants of finite links and the limiting geometry of infinitely iterated links. Our object here is to solve a class of free-group surgery problems, specifically, to construct certain slices for the class of links $\mathscr{C}$ where $D(L) \in \mathscr{C}$ if and only if $D(L)$ is an untwisted Whitehead double of a boundary link L. Unfortunately, $\mathscr{C}$ appears not to be atomic for surgery or s-cobordism. However our technique is quite different from the earlier extension [F4] of simply-connected methods and may prove helpful in establishing the limits of the nonsimply-connected theory.

An n-component link is a topological imbedding $\coprod_{i=1}^{n} S_i^1 \overset{L}{\hookrightarrow} S^3$ of n circles in the 3-sphere. It is slice if and only if there is a topological imbedding $\bar{L}$ (dotted arrow) making the diagram:

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$$\begin{array}{ccc} \coprod_{i=1}^{n} S_i^1 & \xhookrightarrow{L} & S^3 \\ \cap \downarrow \partial & & \cap \downarrow \partial \\ \coprod_{i=1}^{n} B_i^2 & \overset{\bar{L}}{\dashrightarrow} & B^4 \end{array}$$

commute. Furthermore, we always assume that L is flat (i.e., extends to an imbedding of $\coprod_{i=1}^{n} S_i^1 \times B^2$ into S^3) and will require that $\bar{L}$ is also flat (i.e., $\bar{L}$ extends to an imbedding of $\coprod_{i=1}^{n} B_i^2 \times B^2$ which restricts to the extension of L).

An arbitrary 4-dimensional surgery problem can be made an isomorphism over π_1. This done, any standard plane may be represented by an imbedded 2-complex K which is the "capped grope" version (see [F4 and FQ]) of $S^2 \vee S^2$. To remove the homology of the standard plane one should look for a 4-manifold N with $\partial N = \partial$ (reg. neib. $K) = \partial \mathcal{N}$ and $H_*(N;Z) \cong H_*(K;Z)$ for $_* \cong 0,1$ and $H_*(N;Z) \cong 0$ for $_* \cong 2,3$. This is equivalent to looking for a slice (in a homology 4-ball) for a link L whose 0-framed surgery is ∂N. Adding the condition "homotopically ribbon" to N (i.e., $\pi_1(\partial N) \to \pi_1(N)$ is onto) translates into seeking a homotopically ribbon slice in a homotopy (hence standard) 4-ball.

Here is an L which occurs for K, a 2-stage capped groped $(S^2 \vee S^2)$

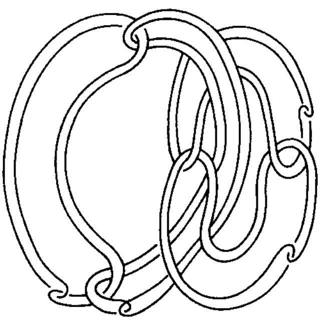


Fig. 1. $L = Wh$ (Bing (Hopf link)), the (untwisted) Whitehead double of the Bing double of the Hopf link

A *boundary link* is defined as the boundary of an oriented surface $S \subset S^3$ which satisfies $\pi_0(\partial S) \underset{\text{inc}_\#}{\cong} \pi_0(S)$. It is easily computed [M] that Bing (Hopf link) is not a boundary link so it does not appear that L belongs to our class and this can actually be proven using the uniqueness of the toroidal decompositions of Haken manifolds [J]. (For fun, find the five characteristic tori in $S^3 - L$, observe that they decompose the link complement into six hyperbolic pieces, two Borromean rings complements and four Whitehead link complements.)

Related examples of links $L \in \mathscr{C}$ can be made, for example, by replacing any two components of Bing (Hopf) by their (possibly twisted) Whitehead doubles and then forming the (untwisted) Whitehead double of the entire resulting link.

A "Whitehead curve" in a framed oriented solid torus is, by definition, isotopic to one of two curves below:

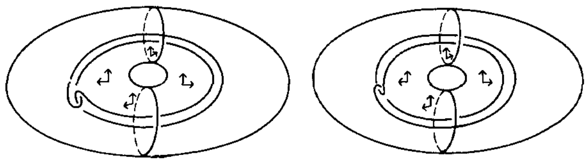


Fig. 2

An (untwisted) Whitehead double of a (tame) simple closed curve in S^3 is formed as follows. First, identify either of the above solid tori with $\mathcal{N}(l)$ so that the framing in the solid torus induces the untwisted normal framing along l and then replace l with the image under the identification of the Whitehead curve. A Whitehead double of a link means a Whitehead double of each of its components. With these definitions we may state our result.

Theorem. *Any link $D(L)$ which is an (untwisted) Whitehead double*[1] *of a tame boundary link L is slice in the sense of bounding a disjoint union of topologically flat disks in B^4. Furthermore, the complement (B^4-disks) is homotopy equivalent to a wedge of circles, with $\pi_1(B^4\text{-disks})$ freely generated by Meridinal loops.*

Proof. The plan is to construct the closed slice complement abstractly as a 4-manifold M^4 satisfying: (1) $M^4 \simeq \vee S^1$'s (2) $\partial M^4 = \mathcal{S}(D(L))$, and (3) $\pi_1(\partial M^4) \to \pi_1(M^4)$ is onto, where $\mathcal{S}$ assigns to a link the 3-manifold which is the result of 0-framed surgery on the link (i.e., cut out a tubular neighborhood of each component and glue it back exchanging meridian and longitude.) Next, 2-handles may be attached to M^4 to "reverse" the earlier surgeries. The result will be a homotopy 4-ball with boundary $= S^3$. By the Poincaré conjecture (theorem) this is B^4; the co-cores of the handles just attached are the (flat) slices in the conclusion of the theorem.

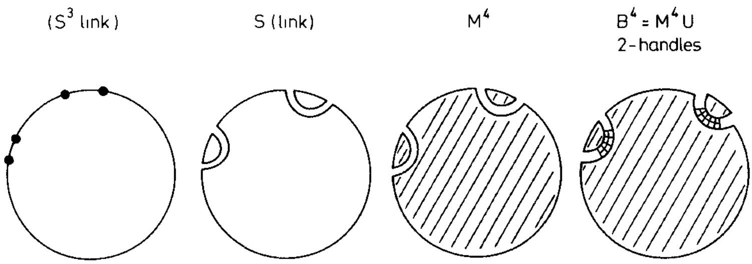


Fig. 3

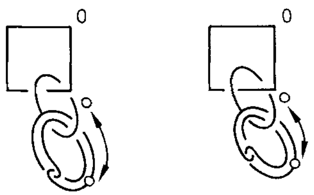
[1] The statement readily implies the same result for "ramified" (untwisted) Whitehead doubling since a ramified doubling agrees with the unramified doubling of the "ramified" link which will still be a boundary link

Recall the notion of a kinky handle [F1]. For simplicity (and without loss of generality), we consider kinky handles k formed from the 2-handle $(D^2 \times D^2, \partial D^2 \times D^2)$ by introducing a single ($+$ or $-$) self plumbing. When discussing the attaching map of a kinky handle there are two possible conventions for measuring framings; we regard the untwisted parallel of $\partial D^2 \times 0 \subset \partial D^2 \times D^2 \subset$ kinky handle to be the one with linking number to $\partial D^2 \times 0$ equal zero (linking as computed by intersection number of spanning disks). This differs from the curve $\partial D^2 \times 0$ by ∓ 2 full twists.

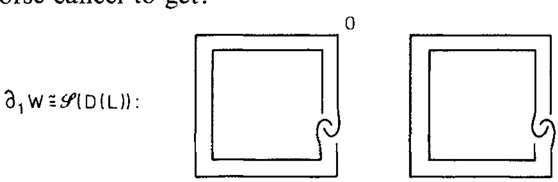
Lemma 1. *Let L be any flat n-component link in S^3. Let $L^\perp$ be the n-component link in $\mathcal{S}(L)$ consisting of the small meridinal linking circles to L. Form $W = \mathcal{S}(L) \times [0,1] \cup (k_1 \cup \ldots \cup k_n)$ by attaching kinky handles to $L^\perp \times 1$ using the zero framing (coming from the inclusion in S^3.) Note that ∂W is in two components $\partial W = \partial_1 W \coprod \partial_0 W$ where $\partial_0 W = \mathcal{S}(L) \times 0$. We claim that $\partial_1 W \cong \mathcal{S}(D(L))$.*

Proof. We use the handle claculus notation [K]. The unknown link is drawn:

on the level of 3-manifolds attaching a kinky handle (see [F3]) yields $\partial_1 W$:



(The dots denote 1-handle, but since we are only looking at the 3-manifold these should not be taken too seriously.) Exchange 0's and dots (see arrows) then Morse cancel to get:



The orientation of the clasps,  or , is governed by the sign of the selfplumbing. This ambiguity is reflected in the fact that L actually has 2^n (usually!) distinct untwisted Whitehead doubles. We will not take the trouble to distinguish them in our notation. □

The next lemma goes back to observations of Milnor [M] and Chen [Ch] in the early 1950's.

Lemma 2. *Let $S \subset S^3$ be a surface which exhibits $L = \partial S$ as a boundary link. Suppose γ is a loop in $S^3 - S$, then (any element in the conjugacy class of) γ is a consequence of a finite number of relations $\{|m_i, m_i^x|\}$ each saying that some meridial circle to a component of L commutes with a conjugate of itself.*

Proof. There is a map $f: S^3 \to S^3$ which is transverse to a collection $\coprod_{i=1}^n D_i^2 \in S^3$ of disks contained in the target and so that $f^{-1}\left(\coprod_{i=1}^n\right) D_i^2 = S$. The loop $f(\gamma)$ is null homotopic in $S^3 - \coprod_{i=1}^n \partial D_i^2$ and is therefore in $\ker(f_\#) \subset \pi = \pi_1(S^3 - L)$.

Stallings theorem relating the lower central series to group homology [S] tells us that $\gamma \in \omega_\infty$, the intersection of the finite portion of the power central series of π.

For any n-component link Q Milnor defines the nilpotent "link group" $G(Q) = \pi_1(S^3 - Q)/\{[m_j, m_j^x]\}$ where the m_j are small meridial loops to components and the m_j^x arbitrary conjugates of m_j. Let Q^i, $i = 1, \ldots, n$, be the link Q with the i^{th} component l_i omitted. It can be checked that the kernel K_i of $G(Q) \xrightarrow{\text{inc}_\#} G(Q_i) \to 0$ is generated by conjugates of m_i and hence is abelian. Furthermore, $K_1, \ldots, K_n$ generate $G(Q)$ so the intersection $\bigcap_{i=1}^n K_i$ is central. Inductively, we now can see that the n^{th} term of the lower central series for the link group $G(Q)$ is trivial. To start the induction, notice that for 1-component links $G \cong Z$. If $\omega_{n-1} G(Q^i)$ is trivial then $\omega_{n-1} G(Q) \subset \bigcap_{i=1}^n K_i \subset$ center $G(Q)$. Consequently, $\omega_n G(Q)$ is trivial.

Combining the last two paragraphs we have $\gamma \in \omega_\infty \subset \omega_n \subset \ker(\pi \to G(\pi))$. The lemma follows. □

Relations of the form $r = [m_i, m_i^x]$ have played a major role in 4-dimensional topology. The finger move (see [C] and [F1]) in which a little patch on a surface is pulled out around a loop "x" and brought back through the surface effectively adds such a relation to the fundamental group of the complement. Although discovered independently by A. Casson and K. Kobayashi ([C and K1] in the early 1970's, the finger move is simply the motion picture of Milnor's fundamental operation in the paper referenced above. In this paper we follow the simple expedient of attatching a (zero-framed) 2-handle, h_i, to a (carefully chosen) loop in $(S^3 - L) \times 0$ which represents the homotopy class of r.

An important notion to our proof is that of "good boundary link". As we have seen any (n-component) boundary link L is inverse to the (n-component) unlink, i.e., there is a degree one map $f: S^3 \to S^3$ with f^{-1} (unlink) $= L$. The homotopy class of $f|: (S^3 - L) \to (S^3 - \text{unlink})$ is sufficiently well defined that L can be called a good boundary[2] link if $f|$ induces an isomorphism on homology with coefficients in $Z[\pi_1(S^3\text{-unlink})]$. On the fundamental group level, $f|$ is giving an epimorphism θ to a free group (which takes a meridian of each component to a distinct generator). The epimorphism counts intersection with an exhibiting Siefert surface S for L.

The homology of the θ-cover $Y = (\widetilde{S^3 - L})^\theta$ can be computed from the Siefert linking matrix of S and in fact formulae for this are given in [F2]. L is a good

[2] This slightly generalizes the definition given in [F2] but does not affect the arguments there

boundary link if and only if $H_1(Y;Z)\cong 0$. A simple (and a-calculational) method for showing $H_1(Y;Z)\cong 0$ is to exhibit some Siefert surface S' for the n-component unlink and an isomorphism of the Siefert forms of S' and S. The computation for $H_1(\widetilde{S^3-\text{unlink}^{\text{univ.}}};Z)$ must come out the same (i.e., trivial) regardless of how S' is chosen and since the answer depends only on the Siefert form, $H_1(\widetilde{S^3-L^\theta};Z)$ will also be trivial.

If L is a good boundary link, the natural map $h\colon \mathscr{S}(L)\to \underset{n\text{-copies}}{\#} S^1\times S^2$ extends to a degree one map on a spin manifold P^4. $\bar{h}\colon P^4\to \underset{n\text{-copies}}{\natural} S^1\times D^3$ with $\partial\bar{h}=h$. (*Proof*: A calculation that the Arf invariants of S vanish implies that $[h]=0\in\Omega_3^{\text{spin}}(\underset{n\text{-copies}}{\bigvee} S^1)$.) Since h is a $Z[\pi_1]$-homology equivalence, $\bar{h}$ is a Wall-surgery problem and $\langle\bar{h}\rangle\in L_4^s(\text{Free})\cong L_4^h\{e\}\cong Z$ is well defined.

The surgery obstruction $\langle\bar{h}\rangle\cong$ (signature P^4)/8. We have observed earlier that $\partial(P^4\cup n(2-\text{handles}))=S^3$, the 2-handles being attached along generators of $H_1(P^4;Z)$. The 3-sphere has Rochlin invariant$=0$. So 16 divides sig $(P^4\cup n(\text{2-handles}))=\operatorname{sig}P^4$. Connected sum with the Kummer surface shows that $\langle\bar{h}\rangle$ is only defined mod 16 and we may therefore assume that $\langle\bar{h}\rangle=0$. (Compare this with the explicit construction in [F2]).

It is possible to do 1-surgeries (see [F4 and FQ]) on P^4 so that the map is an isomorphism on fundamental groups and the surgery kernel is freely generated, as a $Z[\pi_1$ target]-module by a disjoint collection of complexes X_i each one a capped groped-$S^2\vee S^2$. An example of such a thing is drawn below.

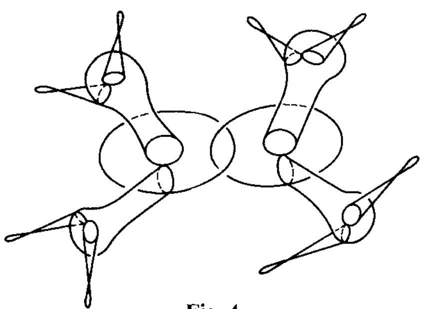


Fig. 4

It is further known ([F3 and F4]) that if a surgery problem's kernel is represented by X_i's as above with the further condition that their inclusions are trivial on π_1 then the surgery problem admits a topological solution, i.e., is topologically normally cobordant to a (topological) homotopy equivalence. Roughly one uses the π_1 information to create a second layer of caps, then a "big embedding theorem" ([F1, F4 or FQ2]) exists to serve as a starting point for infinite constructions. While it is not presently known if all good boundary links are topologically slice, we have:

Lemma 3. *Let L and L' be links in S^3. Let L' be a good boundary link. Suppose there is a 4-manifold Q with $\partial Q=\partial^+Q\amalg\partial^-Q$, $\partial^+Q\cong\mathscr{S}(L)$, $\pi_1(\delta^+Q)\to\pi_1(Q)$ onto, $\partial^-Q\cong\mathscr{S}(L')$, and Q admitting a $Z[\pi_1 C]$ homology equivalence, $h\colon(Q;\partial^+Q,\partial^-Q)\to(C;\partial^+C,\partial^-C)$. The model space is $C=\underset{n\text{-copies}}{\natural} S^1\times D^3-$ interior i*

$(\natural_{k\text{-copies}} S^1 \times D^3)$, *where* i *is an imbedding inducing the trivial map on* π_1, $\partial^+ C = \partial(\natural_{n\text{-copies}} S^1 \times D^3)$ *and* $\partial^- C = i(\partial(\natural_{k\text{-copies}} S^1 \times D^3))$. *Further suppose that* $h_\#\colon \pi_1(Q) \to \pi_1(C)$ *is an isomorphism and that* $h|\partial^- Q$ *is the natural map. Then* L' *is slice in the sense of bounding disjoint flat imbeddings of disks in* B^4. *Furthermore, our construction results in a slice complement* (B^4*-disks*) *which is homotopy equivalent to a wedge of circles as in the conclusion of theorem* 1.

Proof. Let P with $\partial P = \delta(L')$ be the domain of the surgery problem (with the appropriate 1-surgeries completed) constructed above. It is now possible to glue Q and P together along $\partial^- Q \cong \partial P$ to form the domain V of a surgery problem f with target $\natural_{n\text{-copies}} S^1 \times D^3$. It is easy to check that f induces an isomorphism on π_1 and that the two-dimensional kernel (with $Z|\pi_1$ target]-coefficients) is freely generated by the 2-complexes $X_i \subset P$ which maps zero on fundamental group. As remarked above, f may be solved to produce a topological manifold V' with $\partial V' = \partial^+ Q = \mathscr{S}(L')$ and $V' \cong \bigvee_{n\text{-copies}} S^1$. As described earlier, the 4-ball together with the desired slices is now obtained by attaching n 2-handles to V'. $\square$

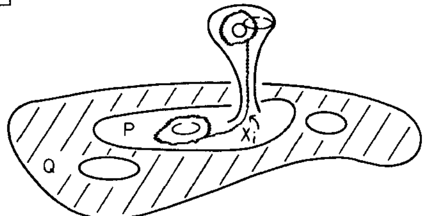


Fig. 5

Let us now return to the manifold W described in Lemma 1. We will form from W a manifold Q satisfying the hypothesis of Lemma 3, with $\partial^+ Q \cong \mathscr{S}(D(L))$ and $\partial^- Q \cong \mathscr{S}(L')$, where L' is some good boundary link which we will describe. This will complete the proof of the theorem. Formation of Q from W occurs in two steps: First, certain 2-handles $\{h_t\}$'s are attached to $\partial^- W$ (with zero framing as measured in $S^3 \times 0$) and second certain (multiply) kinky handles $\{k_i^2\}$, $i \leqq i \leqq n$, (again with framing zero as measured in $S^3 \times 0$) lying in $W \cup$2-handles with attaching region in $\partial^- W$ are deleted.

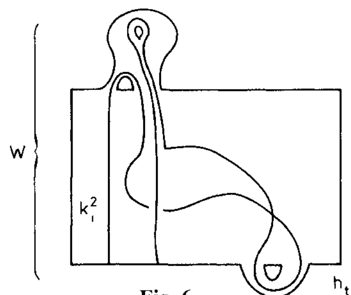


Fig. 6

The picture, which we now describe in detail, is represented schematically in Fig. 6.

There are precursers $\{\dot{k}_i^0\}$, $1 \leqq i \leqq n$, to the $\{\dot{k}_i^2\}$, built from the cores $\dot{k}_i$,

$1 \leqq i \leqq n$, of the kinky handles $k_i \subset W$, $\dot{k}_i^0 = \dot{k}_i \cup \partial \dot{k}_i \times [0,1]$. Dot denotes "core of kinky handle." We have some improvements planned for $\{\dot{k}_i^0\}$ and to effect these we will need three disjoint collections of disjoint-transverse-(immersed)-spheres to $\{\dot{k}_i^0\}$. These are unavailable in W but can be found after the 2-handles h_t are attached.

For each component $l_1, \ldots, l_n$ of L construct three untwisted (linking number$=\mathbf{0}$) parallels l_i^1, l_i^2, l_i^3 in $\mathscr{S}(L)$. For convenience, order these parallels by a single index. We denote them by l^j, $1 \leqq j \leqq 3n$, so $l^{3(i-1)+m} = l_i^m$, $1 \leqq i \leqq n$ and $1 \leqq m \leqq 3$.

The collection of simple closed curves $\{l^j\}$ bound $3n$ disjointly imbedded surfaces in $(S^3 - L) \times 1/2$ formed from three copies of S. Let $\{e_k\}$ be imbedded simple closed curves representing a simplectic basis for $H_1\left(\bigcup_{j=1}^{3n} S^j; Z\right)$. Further, assume $\{e_k\}$ is geometric, i.e., its intersections are as few as possible for a simplectic basis. Each e_k satisfies the hypothesis of Lemma 2 (since each e_k is complementary to a fourth copy of S) so there are a finite number of words, $r_t = [m_{i(t)}, m_{i(t)}^{x_t}] \subset \pi_1((S^3 - L) \times [0,1])$, whose normal closure contains all $[e_k]$. For each word r_t we describe a simple closed curve $\hat{r}_t \subset (S^3 - L) \times 0$ to which the 2-handle h_t is attached.

Find a collection of disjointly imbedded genus-two handle bodies in $(S^3 - L) \times 0$ indexed by t, $\{B_t\}$. Abstractly, $B_t = S^1 \times D^2 \natural I \times D^2 \natural S^1 \times D^2$. Require that the two circles $(S^1 \times 0)$ be small meridial circles, both to the same component of L and that $I \times 0$ be the conjugating arc corresponding (after a choice of base point) to x_t. Inside B_t are many curves representing r_t; below we have made a particular choice for $\hat{r}_t$ which will make L', when defined, a good boundary link.

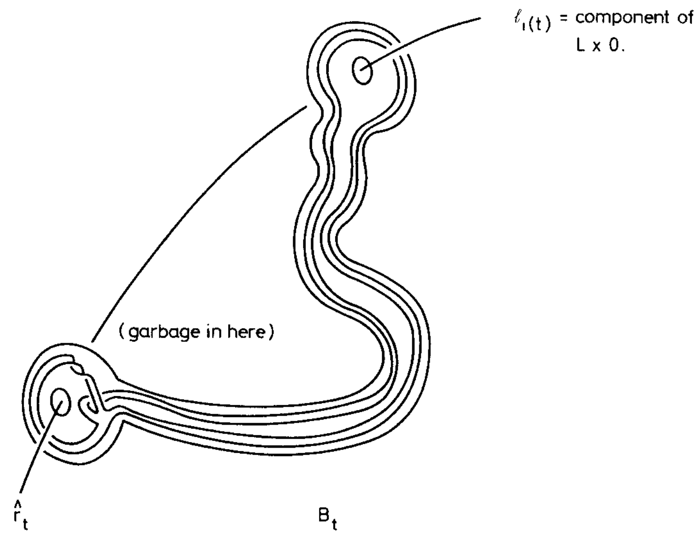


Fig. 7

Notice that because of the small full twist on the left, the obvious genus one Siefert surface for $\hat{r}_t$ visible in Fig. 7, has an "untwisted" simplectic basis (a_t, b_t) satisfying link $(l_{i(t)}, a_t) = 0$, link $(l_{i(t)}, b_t) = 1$ and linking numbers with other l_j, $j \neq i(t)$, all zero. Untwisted means that the normal into the Siefert surface determines a push-off with linking number $=0$. Attach $\{h_t\}$ to $\mathscr{S}(L) \times 0$ along $\{\hat{r}_t\}$ using the zero-framing (induced from S^3).

Now the $\{e_k\}$ bound immersed (and intersecting) disks $\{d_k\}$ in $(S^3 - \text{neib.}(L)) \times [0, 1/2] \cup (\bigcup_t h_t)$. Gluing d_k's to S_j's creates capped surfaces $\{\hat{S}_j\}$ (see [F4 and FQ]) which are disjoint from k_i^0 and intersect each other only in the caps. Hook up these surfaces with copies of the l_i-longitude-spanning disks (arising from the surgeries $S^3 \to \mathscr{S}(L)$) to obtain $\{\hat{\hat{S}}_j\}$, three collections of closed capped surfaces geometrically dual to $\{\dot{k}_i^0\}$ with all intersection within and between collections occurring in the caps.

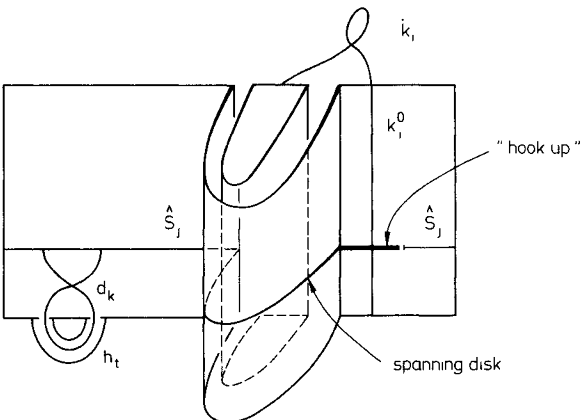


Fig. 8

A process called symmetric surgery in [F4] and contraction in [FQ] exists whereby a closed capped surface $\hat{\hat{S}}$ is cut and glued into an immersed sphere $\tilde{S}$. (Also see [E] for an exposition of this process.) At the cost of introducing new intersections between everything (i.e., other caps, d_k) which cross the "+ caps" and everything which cross the "− caps", disjointness of one $\tilde{S}$ and the other $\tilde{S}$'s is obtained. Contracting one capped surface at a time we finally have three completely disjoint collections of immersed spheres geometrically dual to $\{k_i^0\}$.

Figure 9 should remind the reader of how contraction works.

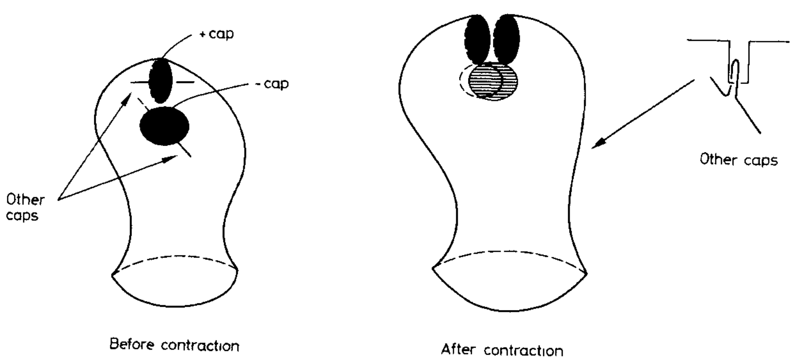


Fig. 9

The double point of each $\dot{k}_i^0$ is removed by piping it into a dual from the first collection (recall Fig. 6). Call the result k_i^1. Now k_i^0 and k_i^1 each has a framing of its attaching regions in $\mathscr{S}(L)\times 0$, coming from $S^3\times 0$; intersection theory in k_i^0 and k_i^1 allows these framings to be identified with an integer. By construction this integer is 0 for k_i^0 and, it is easily computed, will be ± 2 for k_i^1. That is, piping into a framed dual sphere changes the framing by ± 2. An appropriate ($-$ or $+$) connected sum of the cores with a dual from the second collection results in $\{k_i^2\}$ with the framings returned to zero. The k_i^2's have the important new feature that π_1 (image $k_i^2)\to\pi_1(W\cup(\bigcup_t h_t))$ is the zero map. The third collection of duals is used only to control the fundamental group of the complement.

We now define $\bar{W}=W\cup(\bigcup_t h_t)$ and $Q=\bar{W}-\text{interior }\bigcup_{i=1}^n k_i^2)$ with $\partial^+Q=\partial^+W$ and $\partial^-Q=\partial Q-\partial^+Q$. We will show that Q satisfies the hypotheses of Lemma 3 with $L'=D(L)$.

Let us begin by identifying the link L' with $\mathscr{S}(L')=\partial^-Q$. Abstractly, ∂^-Q arises from $S^3\times 0$ in three steps. First, do 0-framed surgery on L; second, do 0-framed surgery on $\{\hat{r}_t\}$; third, delete the solid tori $\amalg\ \partial^-k_i$ and glue in $\amalg\ \partial^+(k_i)$ with zero framing. From the point of view of link calculus (see [K]) only the last step is at all unusual. The prescription for drawing the diagram for this modification is derived in [F1].

Below, L is drawn schematically as two squares. The link diagrams corresponding to our 3-step prescription are also given schematically. (Zero is understood to label any curve representing the attaching region of a 2-handle.)

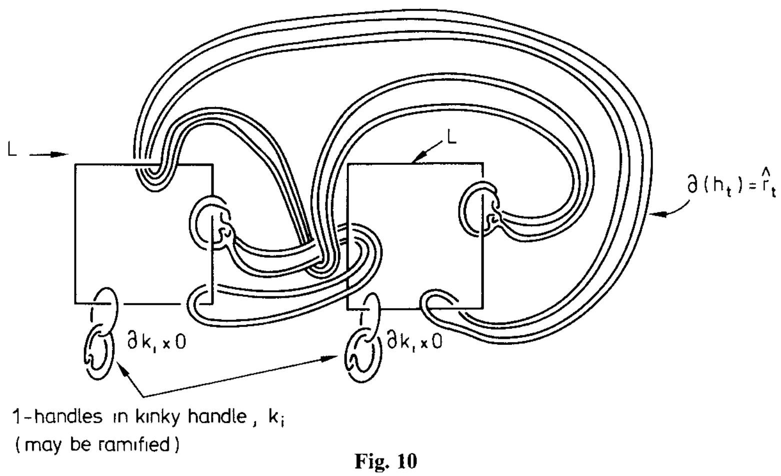


Fig. 10

Trading one and two-handles followed by Morse cancellation leads to a (ramified, untwisted) Whitehead doubling $\bar{D}(L)$ of L with the $\hat{r}_t$-components unchanged. This is the link L':

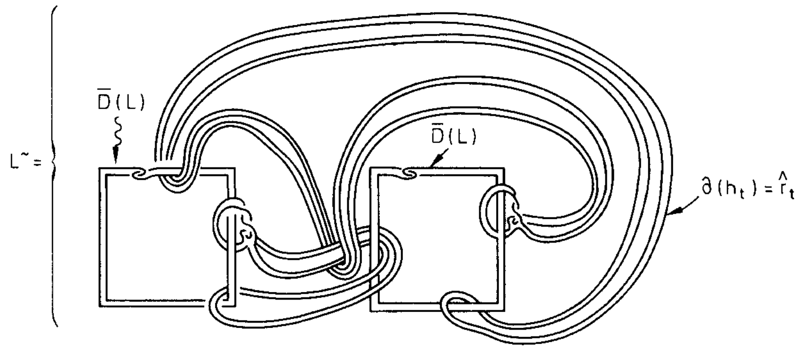


Fig. 11

The point of drawing this schematic is that it makes it easy to verify that all links in the above form are good boundary links. First, pick out the natural bounding surface S'' for L'. The method, as explained earlier, is to find a Siefert surface U for the unlink with the same Siefert matrix as exhibited by S''. This is easily done since very little of the detail of L' is relevant to the Siefert form. Chiefly, one notes that any two components of the original link L have linking number$=0$. For the above example (and only its multiplicities are relevant), the corresponding U is drawn below.

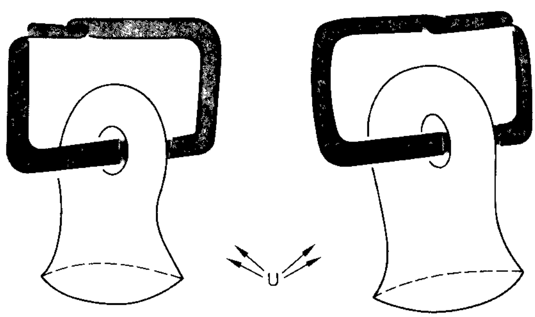


Fig. 12

The proof will be completed by showing that Q has the required homology and fundamental group. To construct the map $h\colon Q \to C$ notice that C contains as proper submanifolds n 3-balls and m 2-balls ($m=$order L') so that cutting along these changes C to $S^3 \times I$. The existence of h (merely) as a degree$=1$ (and normal) map is equivalent to finding a disjoint collection of n 3-manifolds and m surfaces in Q to serve as condidates for f^{-1} (3-balls) and f^{-1} (2-balls). Finding these submanifolds is simply a matter of looking for them; brief directions to aid in their location are given in the two paragraphs which follow.

Consider null homotopies Δ_s in $\overline{W}$ for the natural generating set $\prod_{i=1}^{n} \pi_1(k_i^2)$. These exist since $\pi_1[(\mathscr{S}(L)\times[0,1])\cup(\bigcup_t h_t)] \to \pi_1(\overline{W})$ is the zero map. Let $\{\delta_t\}$ be the co-core disks in the 2-handles $\{h_t\}$. If the Δ_s's and the δ_t's are permitted to cross through $\bigcup_{i=1}^{n} k_i^2$ then the Δ_s's and δ_t's may be taken to be a family of disjoint embeddings. Remove intersections of these null homotopies with $\bigcup_{i=1}^{n} k_i^2$ by piping into copies of the sufaces $\bar{S}_j$, $2n+1 \leqq j \leqq 3n$, underlying the third collection of duals to $\{k^0\}$. Until now, these have been held in reserve. Trimming away collars on the boundary results in a collection of surfaces, $\{\bar{\Delta}_s\} \cup \{\bar{\delta}_t\}$. These are relatively imbedded surfaces in $(Q, \partial^- Q)$ dual to a generating set for $H_1(\partial^- Q; Z)$ and are the surfaces we seek.

Within each kinky handle of W are two "distinguished solid tori" (see [F1]), let T_i be the one whose core is parallel to the tube τ_i which joined the "first collection" to $\dot{k}_i^0$ (in forming $\dot{k}_i^1$.) The T_i are dual to a generating set for $H_1(Q; Z)$ and are a first approximation to the 3-submanifolds we seek. The intersection $T_i \cap \left(\left(\bigcup_{i=1}^{n} k_i^2\right)\right) \cup (\bigcup_s \bar{\Delta} \bigcup_t \bar{\delta})$ consists of the core circle of T_i, where the tube τ_i passes through, parallel copies of the core where the Δ_s's passed through, and small meridial circles linking the original copy of the core where tubes $\{\tau'\}$, arising from the piping to $\{\bar{S}_j, 2n+1 \leqq j \leqq 3n\}$, pass through. These may be eliminated in three steps by bording T_i'''. The core circle intersection is removed, replacing T_i with $T_i' = \partial^+(T_i \times I \cup \mathcal{N}(k_i'))$ a boundary component of a product neighborhood of T_i union a regular neighborhood of the interior component of $k_i^2 - T_i$. The meridial circles are removed by replacing T_i' with $T_i'' = \partial^-(T_i' \times I \cup \mathcal{N}(\text{surface}))$ where these surfaces are those subsurfaces of $(\coprod_s \bar{\Delta}_s) \amalg (\coprod_t \bar{\delta}_t)$ with connected boundary separated by the tubes $\{\tau'\}$. A final bordism based on thickenings of subsurfaces of $(\coprod_s \bar{\Delta}_s) - \coprod_i T_i''$ with connected boundary removes the remaining parallel copies of the core circle. Using $\{T_i''', \ldots, T_n'''\}$ as our collection of 3-submanifolds and the surfaces $\{\bar{\Delta}_s\} \cup \{\bar{\delta}_t\}$ we construct the map h.

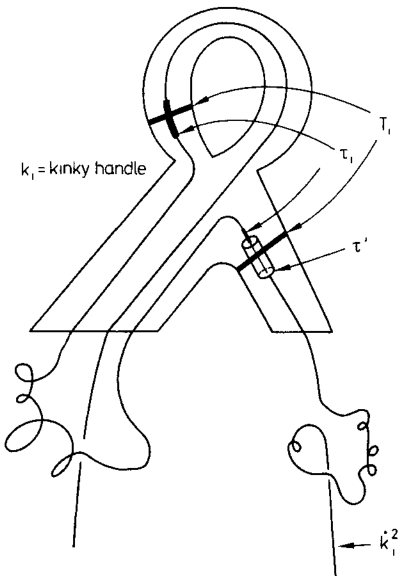


Fig. 13

Since the kinky handles $\{k_1, \ldots, k_n\}$ kill $\pi_1(\mathcal{S}(L))$, it follows from VanKampen's theorem that $\pi_1(\bar{W})$ is a free group with $\{T_i\}$, or for that matter

$\{T_i'''\}$, geometrically dual to a set of generators. Since $\{k_i^2\}$ has a collection of dual spheres (the "third collection"), the inclusion $Q \subset \bar{W}$ is an isomorphism on π_1. But $\{h(T_i''')\} = \{B_i^3\} \subset C$ is geometrically dual to a set of free generators for $\pi_1(C)$, thus h induces an isomorphism $h_\#: \pi_1(Q) \to \pi_1(C)$.

The (untwisted) Whitehead double of a link is a good boundary link if and only if the linking number between any two components is zero. Thus, $D(L)$ is a good boundary link and h restricted to $\partial + Q$ is a $Z[\pi_1(C)]$-homology equivalence. Since the fundamental group of C acts simply by permuting lifts, h restricted to $\partial^- Q$ is a $Z[\pi_1(C)]$-homology equivalence. It follows that the intersection pairing on the kernel K,

$$0 \to K \to H_2(Q; Z[\pi_1(C)]) \to H_2(C; Z[\pi_1(C)]) \to 0$$

is nonsingular, i.e., λ: $K \to \mathrm{Hom}(K, Z[\pi_1(C)]$ is an isomorphism.

It is quite easy to describe the universal cover $\tilde{\bar{W}}^{\text{univ.}} \to \bar{W}$; it is built by plumbing together countably many copies of $\bar{W}'$ where $\bar{W}'$ is $\bar{W}$ after the plumbings in the kinky handles are unidentified. The intersection pairing on $H_2(\bar{W}; Z)$ is identically zero (for this note that link $(\hat{r}_i, \hat{r}_j) = 0$, $\forall\ i, j$). The plumbings do not introduce any nonzero intersections among cycles so the intersection pairing on $H_2(\tilde{\bar{W}}^{\text{univ.}}; Z)$ is also trivial. Since there is an inclusion of universal covers $\tilde{Q}^{\text{univ.}} \subset \tilde{\bar{W}}^{\text{univ.}}$. The intersection pairing must also be identically zero on $H_2(Q; Z[\pi_1(C)])$ so λ is the zero map. Combined with the preceding paragraph, we have $K \cong 0$. Duality for kernel groups (see [W] ch. 2) now implies that h is an equivalence on homology with $Z\pi_1$-coefficients, completing the proof of the theorem. □

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